Thermodynamics of Accelerating Black Holes

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1604:08812 & 1702:00490 [hep-th]
Outline

- What is an accelerating black hole?
- Black Hole Thermodynamics
- Thermodynamics with tension
- Thermodynamics with acceleration
- Charge and critical phenomena
- Outlook
Black hole near SN remnant W44 in Milky Way.

Accelerating a Black Hole?

Black hole is the ultimate slippery object – to accelerate we must be able to push or pull on the actual event horizon.

Anything touching the event horizon must fall in, unless travelling at the speed of light.

Local energy-momentum must have energy=tension.

Luckily – we have a candidate!
Cosmic String?

A cosmic string is a very thin quasi-linear object with an energy momentum that is mainly along the string, an energy equal to tension. Can be formed in field theories with non-simply connected vacua.

\[ \delta = 8\pi G \mu \]

A string produces a conical deficit, but no long range spacetime curvature (no tidal forces).

*e.g. Betti Hartmann’s talk*
The C-Metric

An accelerating black hole is described by the C-metric

\[ ds^2 = \Omega^{-2} \left[ f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 \left( \frac{d\theta^2}{g(\theta)} + g(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} \right) \right] \]

Where

\[ f = \left( 1 - \frac{2m}{r} \right) \left( 1 - A^2 r^2 \right) + \frac{r^2}{\ell^2} \]
\[ g = 1 + 2mA \cos \theta \]
\[ \Omega = 1 + Ar \cos \theta \]

Hong & Teo, CQG20 3629 (2003)
C-Metric Deconstructed

Black hole: parameter “m”, cosmological constant, “l”

\[ ds^2 = \Omega^{-2} \left[ f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 \left( \frac{d\theta^2}{g(\theta)} + g(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} \right) \right] \]

\[ f = \left( 1 - \frac{2m}{r} \right) \left( 1 - A^2 r^2 \right) + \frac{r^2}{\ell^2} \]

\[ g = 1 + 2mA \cos \theta \quad \text{f determines horizon structure -} \]

\[ \Omega = 1 + Ar \cos \theta \quad \text{black hole / acceleration /} \]

\[ \text{cosmological constant} \]
C-Metric Deconstructed

Black hole: Set $A=0$, mass “m”, with (negative) cosmological constant, $-3/\ell^2$

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2}\right)} - r^2 \left(d\theta^2 + \sin^2 \theta \frac{d\phi^2}{K^2}\right)$$

Gives familiar Schwarzschild – AdS

$K$ determines conical singularity on axis / axes
Conical Deficits

Isolating the effect of $K$, look on axis through black hole:

$$ds_{\theta,\phi}^2 \propto d\theta^2 + \frac{\theta^2}{K^2} d\phi^2$$

$K$ relates to tension of “cosmic string” on axis

$$\delta = 2\pi \left(1 - \frac{1}{K}\right) = 8\pi \mu$$

Deconstructing A

Acceleration encoded in “A”

\[ ds^2 = \Omega^{-2} \left[ f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 \left( \frac{d\theta^2}{g(\theta)} + g(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} \right) \right] \]

\[ f = \left( 1 - \frac{2m}{r} \right) \left( 1 - A^2 r^2 \right) + \frac{r^2}{\ell^2} \]

\[ g = 1 + 2mA \cos \theta \]

\[ \Omega = 1 + Ar \cos \theta \]

Acceleration shows up here as a “shift” of infinity, and if m nonzero, a distortion of the spheres at a given “radius”
The Rindler Metric

With no black hole, $A$ appears to modify the AdS length scale, but the conformal factor means the boundary has shifted and is not at infinite $r$

$$ds^2 = \Omega^{-2} \left[ f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]$$

$$f = 1 - A^2 r^2 + \frac{r^2}{\ell^2}$$

$$\Omega = 1 + Ar \cos \theta$$
**Slowly Accelerating Rindler**

For $\mathcal{A} < 1$, there is no horizon, and can change coords:

$$1 + \frac{R^2}{\ell^2} = 1 + (1 - A^2 \ell^2) \frac{r^2}{\ell^2} \quad ; \quad R \sin \Theta = \frac{r \sin \theta}{\Omega}$$

Giving pure AdS!

$$ds^2 = (1 + \frac{R^2}{\ell^2}) dt^2 - \frac{dr^2}{1 + \frac{R^2}{\ell^2}} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$
C-coordinates give AdS from an off-centre perspective. An observer hovering away from centre of AdS is accelerating!

\[ r = 0 \leftrightarrow R = \frac{A^2 \ell^4}{1 - A^2 \ell^2} \]
**Acceleration and Black Hole**

Putting together, A gives the acceleration of the black hole, which appears to lessen the effect of the cosmological constant.

\[ \Lambda_{\text{eff}} = -\frac{3}{\ell^2} \left( 1 - 3A^2\ell^2 \right) \]

The acceleration also gives an imbalance between North and South axes that now have different conical deficits.

- \( \theta \to 0 \)
  \[ ds^2_{\theta,\phi} \propto d\theta^2 + \frac{(1 + 2mA)^2}{K^2} \theta^2 d\phi^2 \]
- \( \theta \to \pi \)
  \[ ds^2_{\theta,\phi} \propto d\theta^2 + \frac{(1 - 2mA)^2}{K^2} (\pi - \theta)^2 d\phi^2 \]
Conical Deficits

Gives different tensions for N and S strings:

\[
\delta_\pm = 2\pi \left( 1 - \frac{g(0)}{K} \right) = 2\pi \left( 1 - \frac{1 \pm 2mA}{K} \right) = "8\pi \mu_\pm"
\]

Conventionally, we make N axis regular, with deficit on S axis

\[
\delta_N = 0 \Rightarrow K = 1 + 2mA
\]

\[
\delta_S = \frac{8\pi mA}{K} \Rightarrow \mu_S = \frac{mA}{K}
\]
The slowly accelerating black hole in AdS is displaced from centre. It has a conical deficit running from the horizon to the boundary. The string tension provides the force that hold the black hole off-centre.
Physical Interpretation

\[ \delta_+ - \delta_- \text{ or } \delta_S = \frac{8\pi mA}{K} \Rightarrow \mu_S = \frac{mA}{K} \]

The tension of the string provides a force to accelerate the black hole, so if we interpret A as the acceleration, then \( M = \frac{m}{K} \) is suggested by Newton’s Law.

\[ \mu = F = MA \]
In general, the C-metric has acceleration horizon, so thermodynamics would refer locally to black hole horizon – here we can see the nontrivial nature of the spacetime more readily.

We look at slowly accelerating black holes to have only 1 horizon.
Black Hole Thermodynamics

Take a look at semi-familiar case: Schwarzschild-AdS with a deficit. Black hole horizon defined by \( f=0 \), look at small changes in \( f \). Horizon still defined by \( f(r) = 0 \).

\[
f(r_+ + \delta r_+) = f'(r_+) \delta r_+ + \frac{\partial f}{\partial m} \delta m + \frac{\partial f}{\partial \lambda} \delta \lambda = 0
\]

Changes \( r_+ \), hence \( S \)

Changes \( m \), hence \( M \)

Changes \( \lambda \), hence \( \Lambda \)
Black Hole Thermodynamics

Temperature has usual definition, but entropy depends on $K$:

$$T = \frac{f'(r_+)}{4\pi}$$

$$S = \frac{\pi r_+^2}{K}$$

So

$$f'(r_+)\delta r_+ = \frac{2K}{r_+} \left( T\delta S + \frac{r_+^2 f'(r_+)}{4} \frac{\delta K}{K^2} \right)$$

And we have to identify changes in $K$

(S): Herdeiro, Kleihaus, Kunz, Radu: 0912:3386 [gr-qc]
Changing Tension

Tension is related to $K$:

$$\mu = \frac{1}{4} \left( 1 - \frac{1}{K} \right)$$

So easily get

$$\delta \mu = \frac{\delta K}{4K^2}$$

Finally

$$P = -\Lambda = \frac{3}{8\pi \ell^2} \quad V = \frac{4\pi r^3}{3K}$$
First Law with Tension

Putting together:

\[ 0 = \frac{2K}{r_+} \left( T \delta S + 2(m - r_+) \delta \mu + V \delta P - \delta \left( \frac{m}{K} \right) \right) \]

So identify

\[ M = \frac{m}{K} \]

Then also get Smarr relation:

\[ M = 2TS - 2PV \]
The term multiplying the variation in tension is a “thermodynamic length”

\[ \lambda = r_+ - m \]

Reinforces interpretation of M as **enthalpy**, if black hole grows, it swallows some string, but has also displaced the same amount of energy from environment.

*Kastor & Traschen: 1207:5415 [hep-th]*
Does it make sense?

Consider a cosmic string interacting with a black hole:

The black hole captures the string, but the string keeps moving and slides off, leaving a portion behind and the black hole heavier.
Black hole must grow:

\[ \delta M + 2m \delta \mu = T \delta S \]

\[ \delta M = 0 \]
\[ \delta r_+ = 0 \]

\[ \delta \mu \approx \mu \]
\[ \delta \mu \approx -\mu \]

\[ \delta m = 4m \delta \mu \]
\[ \delta M = 2m \mu \]

Captures string inside the horizon.
Using the usual Euclidean method, find temperature:

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+^2} \left[ m(1 - A^2 r_+^2) + \frac{r_+^3}{\ell^2(1 - A^2 r_+^2)} \right]$$

But entropy more complicated:

$$S = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}$$
Horizon still defined by $f(r) = 0$:

$$f(r_+ + \delta r_+) = f'(r_+)\delta r_+ + \frac{\partial f}{\partial m}\delta m + \frac{\partial f}{\partial A}\delta A + \frac{\partial f}{\partial \ell}\delta \ell = 0$$

But change in entropy relates to change in horizon radius, tension and $m/K$. Also tension depends on acceleration and $m$ as well as $K$.

$$\mu = \frac{mA}{K}$$
First Law

Keep \( M = m/K \), and define

\[
V = \frac{4\pi r_+^3}{3K(1 - A^2r_+^2)}, \quad \lambda_\pm = \frac{r_+}{1 \pm Ar_+} - KM
\]

To get first law:

\[
\delta M = T\delta S + V\delta P - \lambda_+\delta \mu_+ - \lambda_-\delta \mu_-
\]
Thermodynamic Length

Thermodynamic length for each axis of string.

\[ \lambda_{\pm} = \frac{r^+}{1 \pm Ar^+} - KM \]

Geometric length from pole to singularity, plus compensator.
Two effects:

One, the black hole – small black holes in AdS similar to Schwarzschild, large black holes have positive specific heat.

Two: Acceleration: In flat spacetime generates a temperature (Rindler), here, alters T – but is it similar to lambda?

\[ T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+^2} \left[ m(1 - A^2 r_+^2) + \frac{r_+^3}{\ell^2(1 - A^2 r_+^2)} \right] \]

\[ \frac{1}{\ell^2} - A^2 \simeq \frac{1}{\ell_{\text{eff}}^2} \]
Compare to de Sitter: Lambda pulls the black hole event horizon out and lowers $T$
Acceleration a little more subtle, larger mass black holes have increasing temperature with $A$.
In AdS, large black holes have positive specific heat. Acceleration makes black holes of a given mass more thermodynamically stable.
With Charge:

M is now modified, as is the electrostatic potential:

\[ M = \frac{m}{K(1 + e^2 A^2)} \]

\[ \Phi = \frac{e}{r_+} - \frac{meA^2}{1 + e^2 A^2} \]

\[ \lambda_{\pm} = \frac{r_+}{1 \pm Ar_+} - \frac{m(1 - e^2 A^2)}{(1 + a^2 A^2)^2} \mp \frac{e^2 A}{1 + e^2 A^2} \]

See also Astorino 1612.04387 [gr-qc]
With charge, there is a lower bound on mass from extremal limit. At fixed Q, black holes have positive specific heat near the extremal limit, and at large mass, negative specific heat only for low Q.

Acceleration again makes black more thermodynamically stable.
Phase Structure

At fixed $Q$, see the phase structure more easily by plotting free energy:
Critical T at which mass of preferred black hole jumps – disappears for large accelerations.
In Grand Canonical Ensemble, similar response to acceleration.

\[
G = M - TS - Q\Phi
\]
Summary

- Have shown how to allow for varying tension in thermodynamics of black holes.
- Conjugate variable is *Thermodynamic Length*
- Thermodynamics of accelerating black holes is computable – non-static and non-isolated.
- Acceleration makes black holes more thermodynamically stable.
Boundary Metric

\[ ds^2_b = \left[ 1 - A^2 \ell^2 g(\theta) \sin^2 \theta \right] \frac{dt^2}{\ell^2} - g(\theta) \sin^2 \theta \frac{d\phi^2}{K^2} - \frac{d\theta^2}{g(\theta) \left[ 1 - A^2 \ell^2 g(\theta) \sin^2 \theta \right]} \]